Solving the Faddeev Integral Equation without Two-Body t-Matrices Using the Padé Approximation Technique

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In this paper, we propose a novel revision of the Faddeev equation for three-body scattering. Instead of using two-body transition matrices obtained from the solution of the Lippmann–Schwinger equation, this revised formulation directly incorporates the two-body interactions. The proposed approach is expected to simplify numerical implementation and significantly reduce the computation time required for solving the integral equations using the Padé approximation technique

Faddeev equation for three-body Scattering states

Faddeev equation: $T|\phi\rangle = tP|\phi\rangle + tG_0PT|\phi\rangle,$

2B T-matrices: Lippmann-Schwinger equation

$$t = V + VG_0t$$

Permutation operator

$$P = P_{12}P_{23} + P_{13}P_{23},$$

Free propagator

$$G_0 = (E - H_0 + i\varepsilon)^{-1}$$

Original Faddeev equation in momentum space

$$T|\phi\rangle = tP|\phi\rangle + tG_0PT|\phi\rangle,$$

$$< \vec{p}\vec{q}|T|\phi_d\vec{q_0} > = t_s(\vec{p}, \frac{1}{2}\vec{q} + \vec{q_0}, E - \frac{3q^2}{4m})\phi_d(|\vec{q} + \frac{1}{2}\vec{q_0}|) + \int d^3q'' \frac{t_s(\vec{p}, \frac{1}{2}\vec{q} + \vec{q}'', E - \frac{3q^2}{4m})}{E - \frac{(q''^2 + q^2 + \vec{q}\cdot\vec{q}'')}{m}} < (\vec{q} + \frac{1}{2}\vec{q}'')\vec{q}''|T|\phi_d\vec{q_0} >$$

Extracting the residue of the two-body t-matrices

residue of the two-body t-matrices:

$$< \vec{p}\vec{q}|T|\phi_d\vec{q_0} > = \frac{< \vec{p}\vec{q}|\hat{T}|\phi_d\vec{q_0} >}{\epsilon - E_d}$$

$$t_s(\vec{p}, \vec{q}, E) = \frac{\hat{t}_s(\vec{p}, \vec{q}, E)}{\epsilon - E_d}$$

$$< \vec{p}\vec{q}|\hat{T}|\phi_d\vec{q_0} > = \hat{t}_s(\vec{p}, \frac{1}{2}\vec{q} + \vec{q_0}, E - \frac{3q^2}{4m})\phi_d(|\vec{q} + \frac{1}{2}\vec{q_0}|) + \int d^3q'' \frac{\hat{t}_s(\vec{p}, \frac{1}{2}\vec{q} + \vec{q}'', E - \frac{3q^2}{4m})}{E - \frac{(q''^2 + q^2 + \vec{q}\cdot\vec{q}'')}{m}} \frac{< (\vec{q} + \frac{1}{2}\vec{q}'')\vec{q}''|\hat{T}|\phi_d\vec{q_0} >}{E - \frac{3q''^2}{4m} - E_d + i\epsilon}$$

The new form of Faddeev equation

Representation in momentum space:

$$< \vec{p}\vec{q}|T|\phi_d\vec{q_0} > = V_s(\vec{p}, \frac{1}{2}\vec{q} + \vec{q_0}, E - \frac{3q^2}{4m})\phi_d(|\vec{q} + \frac{1}{2}\vec{q_0}|) + \int d^3q'' \frac{V_s(\vec{p}, \frac{1}{2}\vec{q} + \vec{q}'', E - \frac{3q^2}{4m})}{E - \frac{(q''^2 + q^2 + \vec{q}\cdot\vec{q}'')}{m}} < (\vec{q} + \frac{1}{2}\vec{q}'')\vec{q}''|T|\phi_d\vec{q_0} > + \int d^3p' \frac{V_s(\vec{p}, \vec{p}', E - \frac{3q^2}{4m})}{2(E - \frac{p'^2}{m} - \frac{3q^2}{4m})} < \vec{p}'\vec{q}|T|\phi_d\vec{q_0} >$$

Extracting the residue of the two-body t-matrices

$$V_s(\vec{p}, \vec{q}, E) = \frac{\hat{V}_s(\vec{p}, \vec{q}, E)}{\epsilon - E_d}$$

$$< \vec{p}\vec{q}|T|\phi_d\vec{q_0} > = \frac{< \vec{p}\vec{q}|\hat{T}|\phi_d\vec{q_0} >}{\epsilon - E_d}$$

$$< \vec{p}\vec{q}|\hat{T}|\phi_d\vec{q_0} > = \hat{V}_s(\vec{p}, \frac{1}{2}\vec{q} + \vec{q_0}, E - \frac{3q^2}{4m})\phi_d(|\vec{q} + \frac{1}{2}\vec{q_0}|) + \int d^3q'' \frac{\hat{V}_s(\vec{p}, \frac{1}{2}\vec{q} + \vec{q}'', E - \frac{3q^2}{4m})}{E - \frac{(q''^2 + q^2 + \vec{q}\cdot\vec{q}'')}{m}} \frac{< (\vec{q} + \frac{1}{2}\vec{q}'')\vec{q}''|\hat{T}|\phi_d\vec{q_0} >}{E - \frac{3q''^2}{4m} - E_d + i\epsilon} + \frac{1}{2}\int d^3p' \frac{\hat{V}_s(\vec{p}, \vec{p}', E - \frac{3q^2}{4m})}{E - \frac{p'^2}{m} - \frac{3q^2}{4m}} \frac{< \vec{p}'\vec{q}|\hat{T}|\phi_d\vec{q_0} >}{E - \frac{3q^2}{4m} - E_d + i\epsilon}$$

Numerical tests

Malfliet-Tjon IIIa potential

$$V(\vec{p}, \vec{p}') = \frac{1}{2\pi^2}\left(\frac{V_R}{(\vec{p} - \vec{p}')^2 + \mu_R^2} - \frac{V_A}{(\vec{p} - \vec{p}')^2 + \mu_A^2}\right)$$

$V_R[MeV\,fm]$ 1438.7228

$\mu_A[fm^{-1}]$ 1.550

$V_A[MeV\,fm]$ −626.8932

$\mu_R[fm^{-1}]$ 3.11

A comparison of new and original form of Faddeev equations

Original form

$$< \vec{p}\vec{q}|\hat{T}|\phi_d\vec{q_0} > = \hat{t}_s(\vec{p}, \frac{1}{2}\vec{q} + \vec{q_0}, E - \frac{3q^2}{4m})\phi_d(|\vec{q} + \frac{1}{2}\vec{q_0}|) + \int d^3q'' \frac{\hat{t}_s(\vec{p}, \frac{1}{2}\vec{q} + \vec{q}'', E - \frac{3q^2}{4m})}{E - \frac{(q''^2 + q^2 + \vec{q}\cdot\vec{q}'')}{m}} \frac{< (\vec{q} + \frac{1}{2}\vec{q}'')\vec{q}''|\hat{T}|\phi_d\vec{q_0} >}{E - \frac{3q''^2}{4m} - E_d + i\epsilon}$$

New form:

$$< \vec{p}\vec{q}|\hat{T}|\phi_d\vec{q_0} > - \frac{1}{2}\int d^3p' \frac{\hat{V}_s(\vec{p}, \vec{p}', E - \frac{3q^2}{4m})}{E - \frac{p'^2}{m} - \frac{3q^2}{4m}} \frac{< \vec{p}'\vec{q}|\hat{T}|\phi_d\vec{q_0} >}{E - \frac{3q^2}{4m} - E_d + i\epsilon}$$

$$f(z) \approx \tilde{R}(z) = \frac{A(z)}{B(z)}$$

$$= \hat{V}_s(\vec{p}, \frac{1}{2}\vec{q} + \vec{q_0}, E - \frac{3q^2}{4m})\phi_d(|\vec{q} + \frac{1}{2}\vec{q_0}|) + \int d^3q'' \frac{\hat{V}_s(\vec{p}, \frac{1}{2}\vec{q} + \vec{q}'', E - \frac{3q^2}{4m})}{E - \frac{(q''^2 + q^2 + \vec{q}\cdot\vec{q}'')}{m}} \frac{< (\vec{q} + \frac{1}{2}\vec{q}'')\vec{q}''|\hat{T}|\phi_d\vec{q_0} >}{E - \frac{3q''^2}{4m} - E_d + i\epsilon}$$

- The new form completely avoids the 2B t-matrices,
  - saves the runtime and computational memory,
- The new form has an extra term which is integration on 2B interaction,
  - the extra term in the new form has no interpolations on shifted momenta or angles and no significant computational cost!

Padé Method and The Solution of The Inhomogeneous Equation for Scattering Operator

$$f(z) = a_0 + a_1z + a_2z^2 + \cdots, +a_kz^k + \cdots.$$

$f(z)$ is a meromorphic function near $z = 0$

However, $f(z)$ can be represented everywhere by the quotient of two polynomials $P_N(z)$ and $Q_M(z)$ of degrees N and M, respectively.

$$R_{N,M}(z) = \frac{P_N(z)}{Q_M(Z)} = \frac{P_0 + P_1z + P_2z^2 + ... + P_Nz^N}{1 + q_1z + q_2z^2 + ... + q_Mz^M}$$

$$A_{n+1}(z) = A_n(z) + \alpha_{n+1}zA_{n-1}(z)$$

$$B_{n+1}(z) = B_n(z) + \alpha_{n+1}zB_{n-1}(z)$$

$$\frac{A(0)}{B(0)} = \frac{\alpha_0}{1},$$

$$\frac{A_1(z)}{B_1(z)} = \frac{\alpha_0 + \alpha_1z}{1},$$

$$\frac{A_2(z)}{B_2(z)} = \frac{\alpha_0 + \alpha_1z + \alpha_0\alpha_1z}{1 + \alpha_2z},$$

$\dots,$

$\alpha_0 = a_0$ and $\alpha_1 = a_1,$

Padé Method and The Solution of The Inhomogeneous Equation for Scattering Operator

$$\alpha_{2n} = -\frac{C(n+1/n)C(n-1/n-1)}{C(n/n-1)C(n/n)},$$

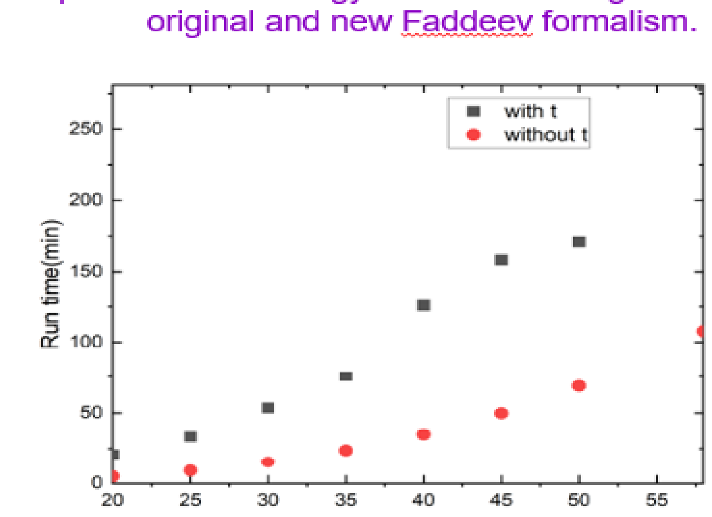
$$\alpha_{2n+1} = -\frac{C(n+1/n+1)C(n/n-1)}{C(n/n)C(n+1/n)}, n \geq 1,$$

$$C(r/s) = det|C| = det\begin{vmatrix} a_{r-s+1} & a_{r-s+2} & \cdots & a_r \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ a_r & a_{r+1} & \cdots & a_{r+s-1} \end{vmatrix}$$

The matrix **C** has the dimension $s \times s$. When $s = 0$, $C(r/s)$ should be set equal to1

Numerical tests: Runtime per energy search

Runtime per search in energy for 3B Scattering calculations using original and new Faddeev formalism.



The new form of Faddeev equations significantly save the computational time again!

References

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